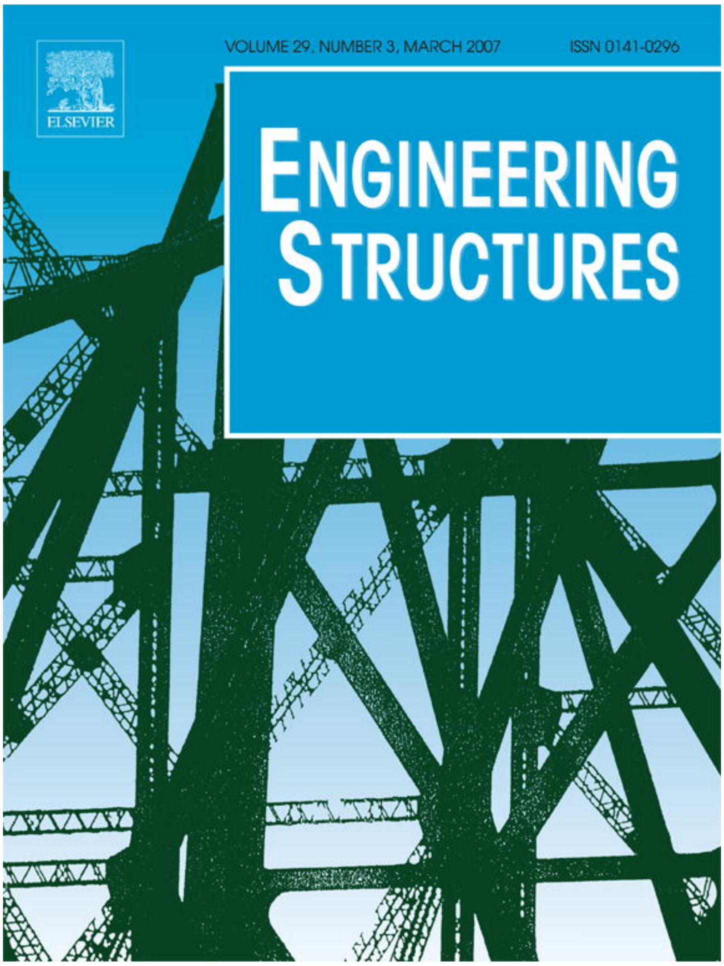
VOLUME 29, NUMBER 3, MARCH 2007
ISSN 0141-0296
ELSEVIER
ENGINEERING
STRUCTURES

Available online at www.sciencedirect.com
ScienceDirect
Engineering Structures 29 (2007) 408–417
ENGINEERING STRUCTURES
www.elsevier.com/locate/engstruct

Compressive behaviour of concrete elliptical columns confined by single hoops

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Received 26 April 2005; received in revised form 10 April 2006; accepted 8 May 2006
Available online 11 July 2006

Abstract

The compressive behaviour of short concrete members having elliptical cross-sections and confined with single steel hoops is investigated.

A simplified model for compressed concrete members with elliptical cross-sections able to determine the maximum compressive strength of confined concrete is proposed. Based on equilibrium consideration the confining pressures due to steel hoops, referring to a fictitious reduced area of the confined core, were determined. Therefore, the stress–train curves of confined concrete are generated by using compatibility conditions and referring to a fictitious equivalent compressed cylinder. Finally analytical results are compared with experimental values available in the literature showing good agreement.

Keywords: Confinement; Hoops; Ellipse; Maximum compressive strength; Ultimate strain; Ultimate stress; Stress–strain curves

1. Introduction

It is well known that if concrete members are confined by transverse steel reinforcement, increases in the bearing capacity and in the corresponding strain are observed experimentally because of the reduced lateral expansion of the concrete core.

A number of stress–strain models and a simplified formula to predict the maximum compressive strength and ultimate strain capacities of compressed concrete members confined by transverse steel have been presented in the past 70 years. Moreover, in the past 15 years interest has also been addressed to the effect of confinement induced by transverse steel on concrete members made of high performance concretes, such as high-strength normal and lightweight types. Some studies [1–10] have focused attention on the principal parameters governing confinement in the concrete core, highlighting the importance of: concrete grade; shape of the transverse cross-section of the members to be reinforced in relation with the type and the grade of transverse and longitudinal steel (hoop, hoops, spirals, ties, jackets, etc.); size of the specimens; loading rate; etc.

Very few experimental and analytical studies are available for compressed members having elliptical cross-sections and confined by single or double hoops, especially referring to the case of high-strength (normal or lightweight) concretes [11, 12], although the topic is of interest for the realization of new structural members such as piers of bridges, or for optimizing the shape to be obtained before jacketing a slender rectangular cross-section.

In the present paper a simplified model will be presented which is able to determine the stress–strain curves of compressed members having elliptical cross-sections confined by steel hoops. Effective stresses along the perimeter of the hoops are evaluated during the loading process including the elastic and plastic phases of behaviour of the transverse steel. The effective volume of the concrete core at rupture is evaluated and on the basis of the maximum confinement pressures the maximum compressive strength is estimated.

The aim of this paper was to develop an analytical model to predict the compressive response of RC members confined with single elliptical hoops and also to give suitable indications for practical applications. Confinement effects due to transverse

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doi:10.1016/j.engstruct.2006.05.006

single hoops were analysed and the proposed model is in accordance with European standards.

Specifically, the present model as described in detail below gives expressions of the effectiveness coefficient k_e and of the geometrical ratio of transverse stirrups, both parameters required, according to Eurocode 8, for ductility estimation. Moreover, the model allows one to predict the maximum bearing capacity and the corresponding strain and also gives information on the ultimate loads and therefore in a global manner on the ductility of compressed members with elliptical cross-sections.

2. Summary of some of the available models

This section provides an overview of some of the existing analytical models referring to compressed concrete members with elliptical cross-sections confined by single hoops or spirals.

The two abovementioned researches were chosen because they are supported by well documented and extensive experimental investigations, and both propose simple analytical models to predict the maximum compressive strength [11,12] and stress–strain curve [12].

2.1. Tan and Yip (1999)

Tan and Yip [11] conducted experiments on eighteen short concrete columns having length 1000 mm reinforced with longitudinal steel and elliptical steel hoops without cover, and tested to failure under monotonic uniaxial compression. The effects of the amount of lateral reinforcement placed at pitch s, expressed in terms of the volumetric ratio ρ_s (defined better below) between 0.6% and 1.8% and the shape of the elliptical hoops were investigated. The concrete strength f'_c measured on the cylinders ranged from 29.4 to 46 MPa and the yield stress f_y of the lateral reinforcement was 454 MPa. Hoops were welded on the overlap portion. The peak strain of the unconfined concrete was approximately 0.0027.

The experimental results highlight the fact that the effectiveness of the confinement diminishes as the aspect ratio (i.e. the ratio between the lengths of the major and minor axes of the ellipse, denoted in the following by $2a$ and $2b$) increases and becomes insignificant (irrespective of the amount of lateral confinement) when $a/b > 2.6$. Moreover, a linear increase was observed by both authors in the strain at peak stress with increases in the volumetric ratio of lateral reinforcement.

The results obtained in [11] highlight the fact that during the loading process and before the peak load is reached the hoops are in the elastic range too, and at this stage the steel strains do not develop uniformly because of the particular shape of the transverse cross-section. When the column reaches the peak load every part of the hoop has yielded. It was also observed that when the peak load is reached, vertical hairline cracks begin to appear and propagate until a diagonal failure plane begins to emerge and the shear propagates in the direction corresponding to the weakest lateral confinement. The shearing action along the failure plane induces significant distortion in the hoops, but none of the hoops (which have already yielded) fractures.

From the theoretical point of view, in [11] a simple empirical analytical expression is proposed for calculation of the maximum compressive strength of confined concrete f'_{cc}, giving best fitting with the experimental results and having the form:

$$\frac{f'_{cc}}{f'_c} = 1 + k_1 \cdot \frac{f'_\ell}{f'_c} \tag{1}$$

where f'_ℓ is the effective confinement pressure, expressed by

$$f'_\ell = \frac{\rho_s \cdot f_y}{2} \cdot \left(1 - \frac{2 \cdot s}{b}\right) \tag{2}$$

k_1 is an effectiveness coefficient able to take the shape effect into account and expressed by:

$$k_1 = 6.779 - 2.645 \cdot \frac{a}{b} \tag{3}$$

and ρ_s the volumetric ratio of hoops having area A_{st} in the pitch s:

$$\rho_s = \frac{4 \cdot A_{st}}{(a + b) \cdot s}. \tag{4}$$

The expression given in Eq. (1) was calibrated to fit the experimental results best; moreover, for $a = b$ it gives $k_1 = 4.1$, this value being in agreement with the results of other studies (see [16]).

2.2. Khaloo et al. (1999)

Khaloo et al. [12] propose an analytical model able to predict the stress–strain response in compression of short concrete members with elliptical cross-sections. The model is based on the experimental data of El-Dash (1994) referring to compressed medium and high strength (normal and lightweight) concrete members confined by single or interlocking double hoops or spirals. The experimental research mentioned analyses the effects of the amount of lateral reinforcement with ρ_s between 0.67% and 2.6%. The concrete strength measured on the cylinders ranged from 46.5 to 102.5 MPa.

It was observed that in the elliptical cross-section the distribution of the lateral confining pressures is not uniform. Moreover, it was verified that for the cases examined the assumption commonly made that, when the concrete reaches its maximum resistance the hoops yield, is only valid for heavily confined columns; a critical value of $\rho_s > 1.5\%$ was suggested.

The model, of a semi-empirical nature, is based on a single fractional equation able to capture the ascending and descending portions of the stress–strain curve but it requires five parameters to be set. In the model it was assumed that the confinement pressure, constant for all strain values, varies along the height and in the transverse cross-section, and the maximum strength and strain capacity are evaluated referring to a yielding condition in the hoops.

The determination of the axial strength of confined concrete f'_{cc} was based on the use of Eq. (1), considering the effective confinement pressure determined with the following expression:

$$f'_\ell = k_s \cdot k_f \cdot \rho_{se} \cdot f_y \tag{5}$$

where k_s is a spacing factor introduced to take into account variability in the distribution of the lateral pressure in the vertical direction, expressed by

$$k_s = 1 - \frac{s - 25}{b} \quad \text{(in SI units with } s > 25 \text{ mm)} \tag{6}$$

k_f is a strength factor which accounts for the change in confining pressure with a change in the ratio of concrete strength to yield strength of the lateral reinforcement,

$$k_f = 1 - \frac{f'_c}{f_y} \tag{7}$$

and ρ_{se} is the equivalent volumetric ratio composed of the two components ρ_{sx} and ρ_{sy}, the latter being the volumetric ratio of the lateral reinforcement to the concrete core in the shorter and longer directions.

Finally, the k_1 coefficient of Eq. (1) is a coefficient depending on the strength of the concrete and on the amount and arrangement of lateral reinforcement, related to a coefficient k_i reflecting the effect of concrete strength on strength enhancement. The coefficient k_1 has the expression

$$k_1 = 3.1 \cdot k_i = 3.1 \cdot \left[2 - \left(\frac{f'_c}{k_s \cdot k_f \cdot \rho_{se} \cdot f_y}\right)^{0.06}\right]. \tag{8}$$

3. Proposed model

The model proposed here for the determination of the stress–strain curve for confined members having elliptical cross-sections and confined by discontinuous hoops is based on a rearranged version of the dated but very effective analytical relationship originally proposed in [13] and also utilised in [14] for concrete members having circular and rectangular cross-sections and confined with transverse steel (hoops, ties).

Moreover, the proposed model, as described below, allows one to determine the effective equivalent confinement pressures, by using equilibrium conditions and referring to members having elliptical cross-section, necessary for the calculus of maximum compressive strength of confined concrete; while it refers to a simplified model considering a fictitious cylindrical member and utilizes compatibility conditions for the determination of the stress–strain curves on confined concrete.

The well-known model proposed in [14] is based on the following stress–strain relationship:

$$\frac{\sigma}{f'_{cc}} = \frac{\beta \cdot \frac{\varepsilon}{\varepsilon_{cc}}}{\beta - 1 + \left(\frac{\varepsilon}{\varepsilon_{cc}}\right)^{\beta}} \tag{9}$$

Table 1
Coefficients A, B obtained by various researches for Eq. (13)

Author	Concrete type		Coefficients A	Coefficients B
Richard et al. [16]	NSC	/	4.1	1
Cusson and Paultre [1]	/	HSC	2.1	0.7
Li and Ansari [17]	/	HSC	2.4305	0.6376
Attard and Setunge [18]	/	HSC	2.254	0.632
Bing et al. [3]	NSC	/	4.6	1
	/	HSC	2.7	1

where f'_{cc} and ε_{cc} are the maximum strength and strain capacity of confined concrete respectively, having the expression:

$$f'_{cc} = f'_c \cdot \left(2.254 \cdot \sqrt{1 + 7.94 \cdot \frac{f'_\ell}{f'_c}} - 2 \cdot \frac{f'_\ell}{f'_c} - 1.254\right) \tag{10}$$

$$\varepsilon_{cc} = \varepsilon_{c0} \cdot \left[1 + 5 \cdot \left(\frac{f'_{cc}}{f'_c} - 1\right)\right] \tag{11}$$

in which ε_{c0} is the maximum strain of the unconfined concrete and β a parameter modelling the slope of the curve, defined as

$$\beta = \frac{E_c}{E_c - \frac{f'_{cc}}{\varepsilon_{cc}}}. \tag{12}$$

This model is able to take into account the phenomenon of increase in the compressive stress and strain induced by the presence of transverse steel inducing passive confinement.

With reference to the confinement effect in high-strength (HSC) normal weight or lightweight concrete, extensive researches have demonstrated that the linear relationship expressed by Eq. (1) with $k_1 = 4.1$ (as suggested in [11]) highly overestimates the strength of the confined core so that non-linear relationships should be adopted; as an alternative it is possible to adopt a linear expression Eq. (1), but assuming a lower value of k_1.

For these reasons several authors suggest not using Eq. (1), but simple nonlinear relations in the form

$$\frac{f'_{cc}}{f'_c} = 1 + A \cdot \left(\frac{f'_\ell}{f'_c}\right)^B. \tag{13}$$

Various values of the coefficients A and B are suggested by several authors to best fit the experimental results, as shown in Table 1.

Comparison between values obtained by the models mentioned using Eq. (13) and referring also to high-strength matrices, not given here for brevity, shows that the expressions suggested in [14] overestimate the maximum increase in compressive strength with variation in the effective confinement pressure, while the expression given in [1] determines more conservative results.

From here on, referring to normal strength concrete (NSC), Eq. (10) will be utilized.

To calculate the maximum compressive strength, a hypothesis generally accepted is that the transverse and the longitudinal steel bars have yielded. The hypothesis that longitudinal steel bars have yielded at peak stress is almost always verified, because the maximum strain of the unconfined concrete is higher than the yielding strain of the longitudinal bars, while the hypothesis of hoops yielding at peak strength is not always verified, especially when high strength matrices and a moderate percentage of transverse steel are utilised. For this reason this second hypothesis leads to an overestimation of the ultimate load of the confined concrete, while correct calculation should take into account the effective stress in the hoops at peak load obtained by step-by-step analysis.

The model given in [14] allows one to determine the stress–strain curve referring to a constant value of confining pressure throughout the loading history. This hypothesis is correct if the steel is in the yield phase, generally occurring at peak load, but not conservative before this stage; therefore it correctly represents the behaviour of confined concrete if the transverse steel yields at peak strength.

In the present paper a numerical procedure is adopted to determine the stress–strain curves of confined concrete. The starting point is the assumption of the Mander et al. (1988) model [14], but referring to a curve intertwining with several Mander curves, each pertaining to a level of confining pressure corresponding to the current axial and lateral strain values.

In particular the procedure is based on the following steps: an initial value of axial shortening ε is assumed; the lateral strain $\varepsilon_\ell = \nu_c \cdot \varepsilon$ is computed assuming a fixed variation law of ν_c with ε; the effective stress in the transverse steel is computed on the basis of the secant elasticity modulus of the concrete core and of the Poisson coefficient; the effective confinement pressure is calculated by considering the average confinement pressure and an effectiveness coefficient (better described in the next sections); the compressive strength of the confined concrete is calculated using Eq. (13); ε_{cc}, β and finally σ are determined through Eqs. (9), (11) and (12); finally, repeating this procedure for all possible values of axial strain the complete stress–strain curve is plotted.

The focus of the procedure described before is the calculation of the effective confinement pressure for each axial strain. This effective confinement pressure will be determined by the product between the average confinement pressure f_ℓ and an effectiveness coefficient k_e, able to consider its non-uniform distribution in plan and along the height of the member, depending on the shape of the transverse cross-section and on the fact that the hoops are discontinuously placed along the height of the member.

The approach followed to determine f_ℓ and k_e is similar to that proposed in [14] and therefore it is based on the assumption that the effective confinement pressure f'_ℓ can be calculated adopting the relation $f'_\ell = k_e \cdot f_\ell$.

The variation law of the ν_c coefficient with the axial strain, in agreement with that proposed in [15], is assumed to be expressed by

$$\nu_c = \nu_0 \cdot \left[1 + 1.38 \cdot \frac{\varepsilon}{\varepsilon_{cu}} - 5.36 \cdot \left(\frac{\varepsilon}{\varepsilon_{cu}}\right)^2 + 8.59 \cdot \left(\frac{\varepsilon}{\varepsilon_{cu}}\right)^3\right] \tag{14}$$

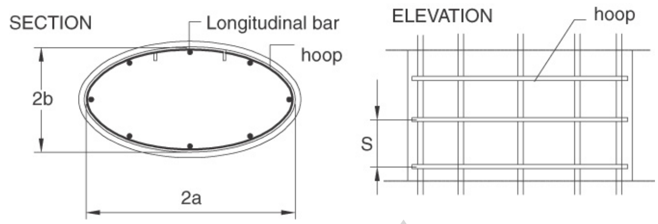

Fig. 1. Typical details of column.

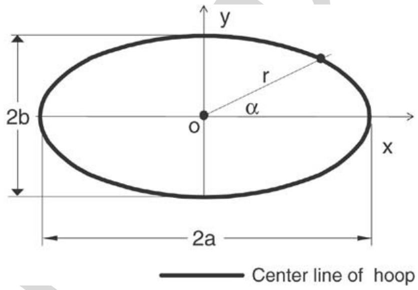

Fig. 2. Definition of ellipse.

ν_0 being the elastic Poisson ratio, assumed to be equal to 0.20, and $\varepsilon_{cu} = 2 \cdot \varepsilon_{c0}$ the ultimate strain of the unconfined concrete. The expressions for confinement pressure and effectiveness coefficient will be determined in the following sections, based on these considerations.

3.1. *Confinement pressures*

The case analysed concerns short concrete members having elliptical cross-sections and reinforced with longitudinal bars and transverse hoops placed at pitch s, as shown in Fig. 1.

Moreover, the model is developed assuming the following simplifications: no (or negligible) cover is considered; size effects are neglected; risk of buckling of longitudinal bars is not considered; and confinement effects induced by longitudinal bars are neglected.

The centre line of the hoops (see Fig. 2) is an ellipse characterized by a minor axis of length $2b$ and a major axis of length $2a$.

If the Cartesian coordinate system has its origin in the centre, the equation of the ellipse, in the polar coordinates, is:

$$r(\alpha) = \sqrt{\frac{a^2 \cdot b^2}{b^2 \cdot \cos^2 \alpha + a^2 \cdot \sin^2 \alpha}}. \tag{15}$$

The area of the ellipse is $\pi \cdot a \cdot b$ and its perimeter is expressed with very good approximation by the following expression:

$$p = \pi \cdot \left[1.5 \cdot (a + b) - \sqrt{a \cdot b}\right] \tag{16}$$

while the volumetric ratio of transverse steel ρ_s is

$$\rho_s = \frac{\left[1.5 \cdot (a + b) - \sqrt{a \cdot b}\right]}{a \cdot b} \cdot \frac{A_{st}}{s}. \tag{17}$$

As shown below, both k_e and f_ℓ depend on the arrangement and mechanical characteristics of the reinforcing jackets and

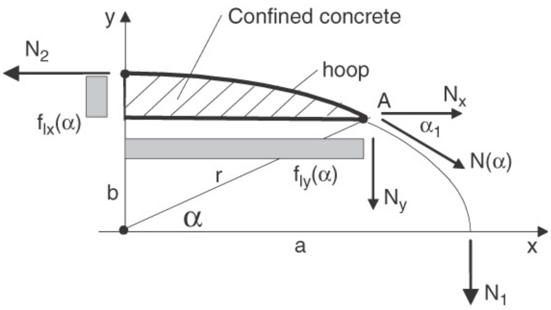

Fig. 3. Equilibrium condition of a portion of the confined cross-section.

on the shape of the transverse cross-section of the reinforced members.

To determine the lateral confinement pressure due to the hoops, it is possible to refer to the equilibrium of a portion of the transverse cross-section, considering the confining hoops as an element having essentially axial stiffness (see Fig. 3).

If the transverse steel is in the elastic range, by imposing the equilibrium conditions of the generic portion of the ellipse corresponding to the polar coordinates r and α and marked with the hatched area in Fig. 3, it is possible to relate the axial force $N(\alpha)$ acting in the hoops with the axial force N_2 in the hoops at the top of the section and the equivalent actions $f_{\ell x}(\alpha)$ and $f_{\ell y}(\alpha)$ due to the presence of concrete.

The abovementioned equilibrium conditions prove to be:

$$f_{\ell y}(\alpha) \cdot x = N_y(\alpha_1) \quad \text{equilibrium in } y \text{ direction} \tag{18}$$

$$N_2 - f_{\ell x}(\alpha) \cdot (b - y) - N_x(\alpha_1) = 0 \quad \text{equilibrium in } x \text{ direction} \tag{19}$$

$$N_2 \cdot (b - y) - f_{\ell x}(\alpha) \cdot \frac{(b - y)^2}{2} - f_{\ell y}(\alpha) \cdot \frac{x^2}{2} = 0 \quad \text{rotational equilibrium (across A).} \tag{20}$$

To correlate the N_x and N_y forces with the N value (see Fig. 3) the angle α_1 was introduced that the tangent forms with the horizontal axis, which is different from α. This angle α_1 is equal to α only in the case of a circle. Considering that the angle α_1 is related to r and α by geometrical relationships (the tangent of α_1 is the prime derivative of the y coordinate) it is possible to obtain the value of $N(\alpha, \alpha_1)$. In the following sections for brevity's sake $N(\alpha, \alpha_1)$ will be indicated as $N(\alpha)$.

Moreover, considering that $N_y(\alpha_1) = N(\alpha_1) \cdot \sin\alpha_1$, $N_x(\alpha_1) = N(\alpha_1) \cdot \cos\alpha_1$ and combining the previous equations, $N(\alpha)$ can be expressed as

$$N(\alpha) = (b - y) \cdot N_2 \cdot \frac{1}{x \cdot \sin\alpha_1 - (b - y) \cdot \cos\alpha_1}. \tag{21}$$

Therefore, considering that $x = r \cdot \cos\alpha$ and $y = r \cdot \operatorname{sen}\alpha$, introducing into Eq. (21) the coordinate r (related to the x and y Cartesian coordinates), one obtains:

$$N(\alpha) = N_2 \cdot \frac{b - r(\alpha) \cdot \sin\alpha}{r(\alpha) \cdot \sin(\alpha + \alpha_1) - b \cdot \cos\alpha_1}. \tag{22}$$

From Eq. (22) it emerges that nonlinear variation in axial force occurs in hoops due to the shape of the transverse cross-section; moreover, for $b = a = R$ (case of a circle) $N_1 = N_2 =$ constant, while for an elliptical cross-section N proves to be variable with α: for $\alpha = 0$, $N_1 = \frac{b}{a} \cdot N_2$, for $\alpha = 90°$, $N_1 = N_2$. Finally, Eq. (22) shows that the maximum axial force occurs along the longer side (being $b/a < 1$).

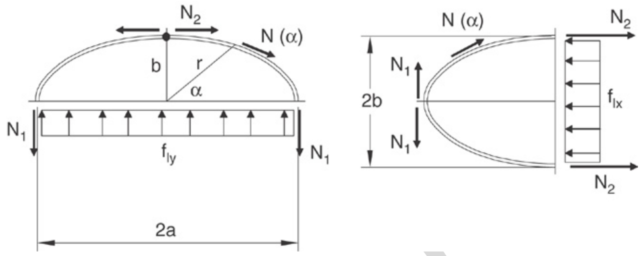

Fig. 4. Equivalent confinement pressures.

It has to be noted that $N(\alpha)$ and consequently the forces N_1 and N_2 depend on both the a and b values and in particular on the ratio a/b, as it is possible to observe from Eq. (22) if it is noted that r is related to a and b.

To determine the equivalent confinement pressures in the x and y directions in the elastic range (see Fig. 4) it appears appropriate to impose the equilibrium conditions of a half-ellipse (separately in the x and y directions).

The confinement pressures, assumed to be distributed in the pitch s (as suggested in [14]), have the expressions:

$$f_{\ell x} = \frac{N_2}{s \cdot b} \tag{23}$$

$$f_{\ell y} = \frac{N_2}{s \cdot b} \cdot \left(\frac{b}{a}\right)^2. \tag{24}$$

From the previous equations it emerges that the ratio between the maximum and minimum axial forces in the hoops is related to the a/b ratio, while the confinement pressures in the x and y directions stay in the ratio $(a/b)^2$.

After first yielding, redistribution of stresses occurs along the hoops until complete yielding occurs and maximum confinement pressures are attained. From simple considerations these values are obtained as

$$f_{\ell x} = \frac{f_y \cdot A_{st}}{s \cdot b} \tag{25}$$

$$f_{\ell y} = \frac{f_y \cdot A_{st}}{s \cdot a}. \tag{26}$$

Through $f_{\ell x}$ and $f_{\ell y}$ it is possible to determine an equivalent pressure f_ℓ, as was already done in [1] for reinforced concrete members with rectangular cross-sections, giving

$$f_\ell = \frac{f_{\ell x} \cdot a + f_{\ell y} \cdot b}{a + b}. \tag{27}$$

This expression, specialized in the elastic range, for the elliptical cross-section gives

$$f_\ell = \frac{N_2}{s} \cdot \frac{\left[\frac{a}{b} + \left(\frac{b}{a}\right)^2\right]}{a + b}. \tag{28}$$

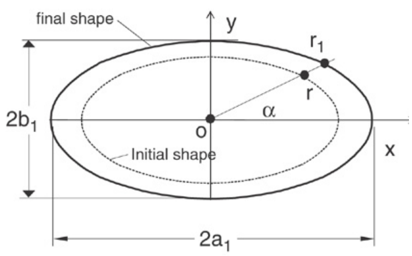

Fig. 5. Lateral expansion of confined cross-section.

At first yielding ($N_2 = f_y \cdot A_{st}$) and when the hoop has completely yielded, f_ℓ takes on the following respective values:

$$f_{\ell(N_2=f_y A_{st})} = \frac{1}{s} \cdot f_y \cdot A_{st} \cdot \frac{\left[\frac{a}{b} + \left(\frac{b}{a}\right)^2\right]}{a+b} \tag{29}$$

$$f_{\ell(N_2=N_1=f_y A_{st})} = \frac{1}{s} \cdot f_y \cdot A_{st} \cdot \frac{\left[\frac{a}{b} + \frac{b}{a}\right]}{a+b}. \tag{30}$$

The difference between these two values is significant with increases in the a/b ratio.

An approximate way to determine the force distribution in the hoops and the confinement pressure at the hoop level for each axial shortening is that based on the definition of an equivalent cylinder better described below. At first, the case of hoops in the elastic range will be examined; then the case of progressive plasticization will be dealt with.

The model based on compatibility principals is simplified and it assumes the following: occurrence of homothetic deformation, meaning that the variation in the perimeter of the ellipse generates constant strain at each step (the latter utilised to determine the lateral strain); and an approximation to determine the lateral expansion of the ellipse considering an equivalent cylinder, in such a way that the expression of the lateral contraction can be assumed as the radial displacement w of the cylinder.

Moreover, it has to be considered that the confinement effects are due to: hoop stress, turnaround forces related to the curvature of the ellipse, etc. In a simplified way these effects were related in a first step to the turnaround forces, which take into account of the ellipse curvature, and therefore the effective force distribution was determined by considering equilibrium considerations also including elastic or plastic behaviour of stirrups depending on the average strain reached.

With reference to Fig. 5, it is supposed that homothetic expansion of the transverse cross-section occurs during the axial shortening process of the concrete members. The free dilatation of the transverse cross-section, measured by the variation in the perimeter of the ellipse, for compatibility between transverse steel and concrete core, is equal to the elongation of the hoop and to the contraction of the transverse cross-section due to the confinement pressures.

For simplicity the transversal deformation along the x and y axes is assumed to be equal to ε_t, the latter being related to ν_c and ε by $\varepsilon_t = \nu_c \cdot \varepsilon$.

At each step this deformation produces variation Δp in the perimeter of the ellipse, which increases from the initial value p up to the final value p_1.

Up to first yielding ($N_2 = f_y \cdot A_{st}$), by imposing the condition that the variation in the perimeter of the ellipse due to ε_t is equal to the elongation of the hoops and to the contraction of the transverse cross-section, one obtains:

$$\Delta p = \int_s \frac{N(s) \cdot ds}{E_s \cdot A_{st}} + w(r,\alpha) = \frac{4 \cdot N_2}{E_s \cdot A_{st}} \cdot \int_0^{\pi/2} \frac{b - r(\alpha) \cdot \sin\alpha}{r(\alpha) \cdot \sin(\alpha + \alpha_1) - b \cdot \cos\alpha_1} \cdot r(\alpha) \cdot d\alpha + w(r,\alpha) \tag{31}$$

where, as already said, $\Delta p = p_1 - p$ and $w(r,\alpha)$ is the reduction in the area of the transverse cross-section related to the compressibility of the concrete, better defined below.

As already mentioned in the previous section, to calculate the confinement pressures at each lateral strain step the ellipse was approximated to an equivalent cylinder. For a cylinder w is the lateral contraction (it is the radial displacement induced by the confinement pressures) while for an ellipse w is the lateral contraction due to the confinement pressures and it is rigorously a function of r, α, and α_1, but in an approximate way it was assumed to only depend on the average radius of the ellipse as in a fictitious cylinder.

The perimeter of the ellipse after the deformation is expressed by

$$p_1 = \pi \cdot \left[1.5 \cdot (a_1 + b_1) - \sqrt{a_1 \cdot b_1}\right] \tag{32}$$

where

$$a_1 = a \cdot (1 + \varepsilon_t) \quad \text{and} \quad b_1 = b \cdot (1 + \varepsilon_t). \tag{33}$$

The definition of $w(r,\alpha)$ is very complex due to the shape of the transverse cross-section and the non-uniform distribution of the confinement pressures. In an approximate way the problem is approached by referring to an equivalent cylinder having radius $R = (a+b)/2$ and subjected to an equivalent uniform pressure f_ℓ defined as in Eq. (28).

With reference to this cylindrical equivalent cross-section, the value $w(r,\alpha)$ proves to be constant for the particular shape of the section; moreover, since the axial stiffness k_r of the cylinder of radius $r = R$ is expressed by:

$$k_r = \frac{E_c}{R \cdot (1 - 2 \cdot \nu_c)} \tag{34}$$

by relating the force $N_2 = N_1 = N$ with the internal pressure by means of equilibrium conditions, one obtains:

$$w = N \cdot \frac{(1 - 2 \cdot \nu_c)}{E_c}. \tag{35}$$

By substituting Eq. (35) into Eq. (31) and considering that $\Delta p = 2 \cdot \pi \cdot R \cdot \varepsilon_t$, one obtains:

$$2 \cdot \pi \cdot R \cdot \varepsilon_t = 4 \cdot \frac{N}{E_s \cdot A_{st}} \cdot \frac{\pi}{2} \cdot R + N \cdot \frac{(1 - 2 \cdot \nu_c)}{E_c}. \tag{36}$$

Therefore, the force N is expressed by

$$N = \frac{2 \cdot \pi \cdot \varepsilon_t}{\frac{2 \cdot \pi}{E_s \cdot A_{st}} + \frac{(1 - 2 \cdot \nu_c)}{R \cdot E_c}}. \tag{37}$$

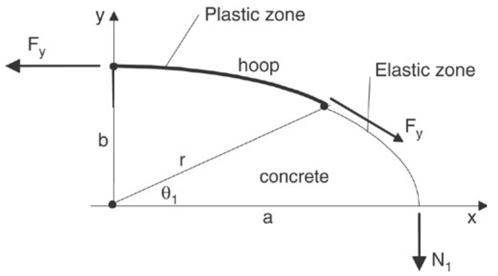

Fig. 6. Equilibrium condition of a portion of the confined section.

In the case of the ellipse, if the integral of Eq. (31) is denoted as $F(a, b)$, also substituting Eq. (15) into (31), one obtains

$$F(a,b) = \int_0^{\pi/2}\left(\frac{b-\sqrt{\frac{a^2\cdot b^2}{b^2\cdot\cos^2\alpha+a^2\cdot\sin^2\alpha}}\cdot\sin\alpha}{\sqrt{\frac{a^2\cdot b^2}{b^2\cdot\cos^2\alpha+a^2\cdot\sin^2\alpha}}\cdot\sin(\alpha+\alpha_1)-b\cdot\cos\alpha_1}\right)\cdot\sqrt{\frac{a^2\cdot b^2}{b^2\cdot\cos^2\alpha+a^2\cdot\sin^2\alpha}}\cdot \mathrm{d}\alpha. \tag{38}$$

By solving Eq. (38) in a closed form, a nonlinear variation in F with a/b is observed and a complex expression is derived. Nevertheless, by plotting Eq. (38) solved numerically, it appears that it can be easily fitted by the following expression:

$$F(a,b) = a\cdot\frac{\pi}{2}\cdot\left(\frac{b}{a}\right)^{1.5}. \tag{39}$$

Moreover, adapting Eq. (35) to the case of the ellipse, one obtains

$$w(r,\alpha) = f_\ell(N_2)\cdot\frac{a+b}{2}\cdot\frac{(1-2\cdot\nu_c)}{E_c}. \tag{40}$$

Replacing Eq. (28) into Eq. (40) and Eqs. (39), (40) into Eq. (31) one obtains

$$\Delta p = 2\pi\cdot a\cdot\frac{N_2}{E_s\cdot A_{st}}\cdot\left(\frac{b}{a}\right)^{1.5} + \frac{N_2}{s}\cdot\frac{\left[\frac{a}{b}+\left(\frac{b}{a}\right)^2\right]}{a+b}\cdot\frac{(a+b)\cdot(1-2\cdot\nu_c)}{2\cdot E_c}. \tag{41}$$

Therefore the force N_2 proves to be:

$$N_2 = \frac{\varepsilon_t\cdot\pi\cdot\left[1.5\cdot(a+b)-\sqrt{a\cdot b}\right]}{\frac{2\pi\cdot a}{E_s\cdot A_{st}}\cdot\left(\frac{b}{a}\right)^{1.5}+\frac{(1-2\cdot\nu_c)}{2\cdot s\cdot E_c}\cdot\left[\frac{a}{b}+\left(\frac{b}{a}\right)^2\right]}$$
$$N_2 \le f_y\cdot A_{st}. \tag{42}$$

If the hoop has partially yielded, it means (as shown in Fig. 6) that an angle θ_1 divides the elastic portion of the hoops from the yielded part. This angle can be derived by setting $\alpha = \theta_1$ and $N(\theta_1) = F_y = f_y\cdot A_{st}$.

At this stage, since $N_2 = N_1\cdot\frac{a}{b}$, Eq. (22) gives the N_1 value (see Fig. 6) in the form

$$N_1 = F_y\cdot\frac{b}{a}\cdot\frac{b-r(\theta_1)\cdot\sin\theta_1}{r(\theta_1)\cdot\sin(\theta_1+\alpha_1)-b\cdot\cos\alpha_1} \tag{43}$$

and therefore, for the elastic zone, one obtains

$$N(\alpha) = F_y\cdot\frac{b-r(\theta_1)\cdot\sin\theta_1}{r(\theta_1)\cdot\sin(\theta_1+\alpha_1)-b\cdot\cos\alpha_1}\cdot\frac{b-r(\alpha)\cdot\sin\alpha}{r(\alpha)\sin(\alpha+\alpha_1)-b\cdot\cos\alpha_1}\quad 0\le\alpha\le\theta_1. \tag{44}$$

By imposing the compatibility conditions, and assuming a linear hardening behaviour governed by the strain-hardening modulus E_h, Eq. (31) becomes:

$$\Delta p = \frac{4}{A_{st}\cdot E_s}\cdot\int_0^{\theta_1}F_y\cdot\frac{b-r(\theta_1)\cdot\sin\theta_1}{r(\theta_1)\cdot\sin(\theta_1+\alpha_1)-b\cdot\cos\alpha_1}\cdot\frac{b-r(\alpha)\cdot\sin\alpha}{r(\alpha)\sin(\alpha+\alpha_1)-b\cdot\cos\alpha_1}\cdot r(\alpha)\cdot\mathrm{d}\alpha + \frac{4}{A_{st}\cdot E_h}\cdot\int_{\theta_1}^{\pi/2}A\cdot F_y\cdot r\cdot\mathrm{d}\alpha + w(r,\alpha) \tag{45}$$

in which A is a coefficient taking the strain hardening effect into account. For each ε_t value Eq. (45) gives the θ_1 value and the corresponding variation in stress in the hoops. It can be utilised up to the ultimate strain of hoops, which has to be considered known. It must be noted that $w(r,\alpha)$ has to be determined with reference to the confinement pressure relative to the state of Fig. 6; therefore, it can be calculated considering the following expression of the confinement pressures:

$$f_{\ell x} = \frac{f_y\cdot A_{st}}{s\cdot b} \tag{46}$$

$$f_{\ell y} = \frac{b}{a}\cdot\frac{b-r(\theta_1)\cdot\sin\theta_1}{r(\theta_1)\cdot\sin(\theta_1+\alpha_1)-b\cdot\cos\alpha_1}\cdot\frac{f_y\cdot A_{st}}{s\cdot a}. \tag{47}$$

These expressions are derived from Eqs. (29) and (39) in which $N_1 = f_y\cdot A_{st}$ and N_1 is expressed by Eq. (43). Determining for each axial shortening the uniform confinement pressure in the x and y directions (Eqs. (46), (47)) and by Eq. (28) the equivalent pressure the effective confinement pressure can be calculated if an adequate expression of the reduction coefficient k_e is derived.

3.2. Effectiveness coefficient k_e

It has to be observed that, as also suggested in [14], in the case of a circular cross-section almost the whole area of the concrete core is effectively confined ($k_e = 1$ for steel spirals with close pitch) and the confinement pressures are almost uniform, while in a square or rectangular cross-section with single hoops non-uniform confinement pressures occur.

In the case of an elliptical cross-section with hoops and without transverse ties, as already observed in the previous section, either when the hoops are in the elastic range or have partially (or completely) yielded, the confinement pressures in the x and y directions are expected to be quite different, especially when the section is slender.

In accordance with the well-known model given in [14] it appears reasonable for this reason to fictitiously reduce the area of the ellipse and to refer to uniform confinement pressures in the x and y directions. The area of the ellipse is reduced by

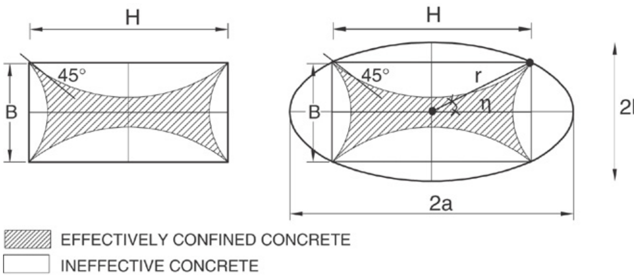

Fig. 7. Effective confined area in rectangular and elliptical transverse cross-sections.

assuming the shape in Fig. 7. A similar shape of the effectively confined concrete was also assumed in [12].

To define this area, a fictitious rectangle with sides B and H is considered, internal to the elliptical cross-section, satisfying the condition that $r = \frac{1}{2} \cdot \sqrt{H^2 + B^2}$.

This rectangle defines an ineffective area (area enclosed between the ellipse and that detached in Fig. 7), which, if $r = a = b$ (case of a circle), determines $k_e = 1$, while it approaches zero when a tends to infinity.

Based on these considerations, the effectively confined area at rupture is that shown in Fig. 7, where the area of the rectangle is penalized by subtracting the areas enclosed in the parabolas having initial tangent 45°.

By calculating the area of the effectively confined concrete core in a simplified way, one obtains

$$k_e \cong \frac{\pi \cdot a \cdot b - \left[\frac{4}{3} \cdot B \cdot \left(a - \frac{H}{2}\right) + \frac{4}{3} \cdot H \cdot \left(b - \frac{B}{2}\right) + \frac{1}{3} \cdot \left(H^2 + B^2\right)\right]}{\pi \cdot a \cdot b}. \tag{48}$$

By expressing B and H by means of r and α (see Fig. 7), Eq. (48) becomes:

$$k_e = 1 - \frac{\frac{8}{3} \cdot r \cdot \text{sen}(\eta) \cdot [a - r \cdot \cos(\eta)] + \frac{8}{3} \cdot r \cdot \cos(\eta) \cdot [b - r \cdot \text{sen}(\eta)]}{\pi \cdot a \cdot b}. \tag{49}$$

The angle η depends on the b/a ratio and it is set in such a way as to give, through Eq. (49), $k_e = 1$ when $a = b = R$, and $k_e = 0$ when $a = 2.6$; this maximum a/b ratio was suggested by [11] as the limit for effectiveness of transverse steel in extended cross-sections.

To obtain these results, it is possible to assume a linear variation in η with the b/a ratio, as follows:

$$\eta = 138° \cdot \frac{b}{a} + 25°. \tag{50}$$

As observed experimentally by Tan and Yip [11] and also by Teng and Lam [19], in a similar problem (compressed elliptical columns confined externally by FRP wraps) when the ratio a/b is reduced the effectively confined core is approximately coincident with the full area and a lower limit can be found in a value of $a/b = 1.25$, while for a slender section when the ratio a/b exceeds the value of 2.6 the effectively confined core can be assumed to be negligible, which means that no confinement effects are expected. In the current model the k_e coefficient reflects the above mentioned aspects; in fact for a circular cross-section it gives $k_e = 1$ and for $a/b = 1.25 k_e = 0.89$, while for $a/b = 2.6$ the k_e value is negligible.

A further reduction in effectively confined concrete is also assumed along the height, because the hoops are placed at pitch s; therefore, the confinement pressures determined by the analysis of the transverse cross-sections will be reduced by k_e and by a coefficient k able to take discontinuous hoops into account. As suggested in [11], k is expressed by

$$k = 1 - \frac{s}{2 \cdot b} \tag{51}$$

and finally the effective confinement pressure is

$$f'_\ell = k \cdot k_e \cdot f_\ell. \tag{52}$$

4. Comparison with experimental results

In this section a comparison between the experimental results obtained by Tan and Yip [11] and analytical values obtained with the present model is carried out. Table 2 gives the geometrical and material properties, and maximum experimental and theoretical compressive stress and strain capacity of confined specimens having elliptical cross-sections and confined by single hoops. Analytical values are obtained assuming for prediction of the maximum compressive strength using Eq. (10) and the models proposed in [11,12]. The experimental researches mentioned here have already been introduced in a previous section of the paper (see Section 2.2). The comparison shows the ability of the model to accurately predict the strength enhancement due to confinement effect considering the effects of the cross-section shape and the mechanical properties of the constituent materials. Maximum compressive strength values obtained by using the expression of the effective confinement pressures given in the present paper (see Eq. (52)) are also in good agreement with analytical values obtained by using the other models (see [11,12]). The use of the proposed approach offers some additional advantages with respect to the others proposed in the literature referring to this particular case. First of all, the confinement pressures f_{lx}, f_{ly} (see Eqs. (23)–(26)) are obtained by using equilibrium conditions able to take into account the particular shape of the cross-section; secondly, the introduction of an effectively confined core by means of k_e coefficient (see Eq. (49)) reflects the aspect that the confinement pressures can be quite different in the x and y directions for slender cross-section, as observed experimentally, and it is a simple and well known approach utilised in the literature for confinement problems (see [14]).

Fig. 8 refers to experimental data given by Tan and Yip [11] for the cases of $a/b = 1.25$ and 2.5 for $\rho_s = 0.6\%$ and 1.8%, and $f'_c = 26.82$ and 21.2 MPa. The values of σ (stresses) are calculated with reference to the confined concrete core. In the same graph analytical curves obtained by using the proposed model are given, showing good agreement. From the analytical results it emerges, as already observed experimentally in [11], that an increase in the volumetric ratio ρ_s increases the

Table 2
Summary of reinforcement, material properties and mechanical properties of tested columns [11]

f'_c (MPa)	a (mm)	b (mm)	a/b	s (mm)	ρ_s (%)	f'_{cc} (MPa)			
						Exp.	Predicted		
						[11]	Current.	[11]	[12]
26.82	227	182	1.25	133	0.6	29.46	30.49	29.70	27.95
26.82	249	166	1.50	136	0.6	30.58	29.07	28.90	27.68
26.82	269	154	1.75	140	0.6	27.22	27.73	28.22	27.41
26.82	288	144	2.00	145	0.6	28.43	28.12	27.65	27.16
26.82	305	136	2.25	150	0.6	26.82	26.90	27.21	26.96
26.82	322	129	2.50	155	0.6	27.43	27.17	26.88	26.82
27.85	227	182	1.25	67	1.2	35.83	35.46	35.20	32.63
27.85	249	166	1.50	68	1.2	34.03	32.87	33.46	32.26
27.85	269	154	1.75	70	1.2	32.54	31.38	31.83	31.79
27.85	288	144	2.00	72	1.2	30.86	29.41	30.37	31.35
27.85	305	136	2.25	75	1.2	29.49	29.15	29.14	30.87
27.85	322	129	2.50	77	1.2	27.85	27.85	28.07	30.51
21.20	227	182	1.25	44	1.8	33.09	33.96	33.27	30.43
21.20	249	166	1.50	45	1.8	32.05	32.20	30.43	29.83
21.20	269	154	1.75	47	1.8	28.12	27.25	27.70	28.97
21.20	288	144	2.00	48	1.8	27.28	26.34	25.41	28.39
21.20	305	136	2.25	50	1.8	23.88	23.34	23.37	27.65
21.20	322	129	2.50	52	1.8	21.46	22.27	21.57	26.97

[11] $f_y = 454$ MPa of hoop; $f_y = 474$ MPa of longitudinal bar.

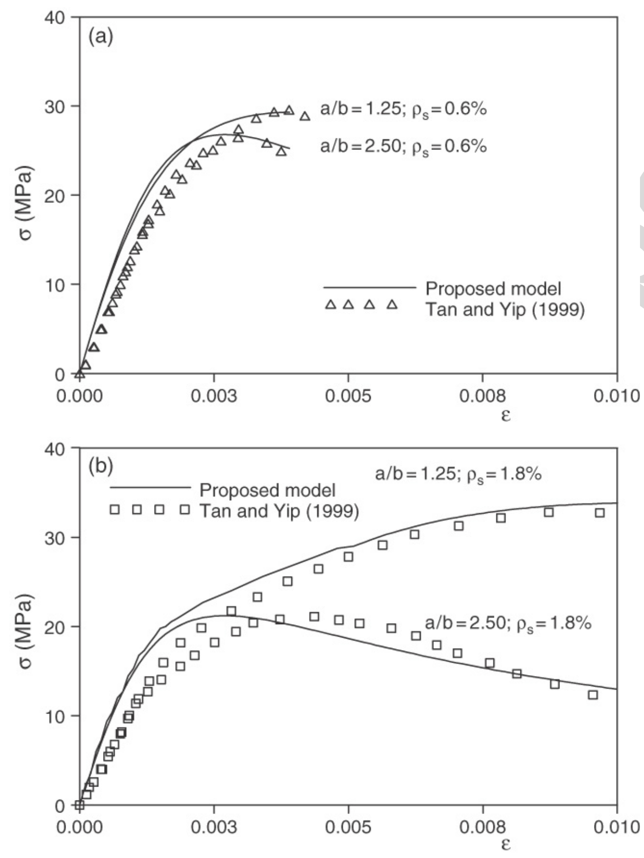

Fig. 8. Comparison between analytical and experimental stress–strain curves. (a) Low volumetric ratio ($\rho_s = 0.6\%$); (b) high volumetric ratio ($\rho_s = 1.8\%$).

peak strength and strain values and the post-peak ductility is enhanced too, while the increase in the ratio a/b reduces strength and ductility. Analytical prediction of the experimental

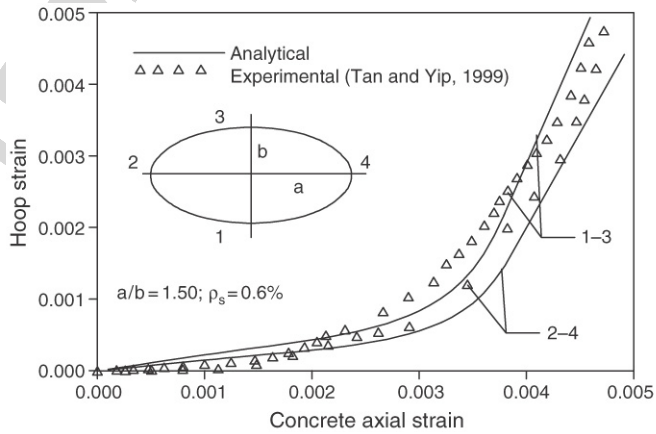

Fig. 9. Experimental and analytical variation in hoop strain with concrete axial strain.

stress–strain curves with the other models mentioned [11,12] are not given here for brevity and because results generated by using the model proposed in [11], as was shown in the original paper, fit very well its reference data, while the model proposed in [12] is able only to predict the peak stress values.

Fig. 9 shows the variation in hoop strain with the axial strain measured experimentally by Tan and Yip [11]. In the same graph analytical values predicted by means of the proposed model are given too. As observed experimentally and reproduced analytically, the steel strains do not develop uniformly: for axial strain lower than yielding value, a quasi-linear variation in the hoop strain is observed, while beyond this stage a significant increase in the hoop strain occurs due to the lateral expansion of the concrete core.

Fig. 10 shows, for the case of $a/b = 1$, 1.25 and 2.5 and ρ_s 1.8%, the analytical dimensionless N_1 force (dimensionless with respect to F_y) with the variation of the plasticization

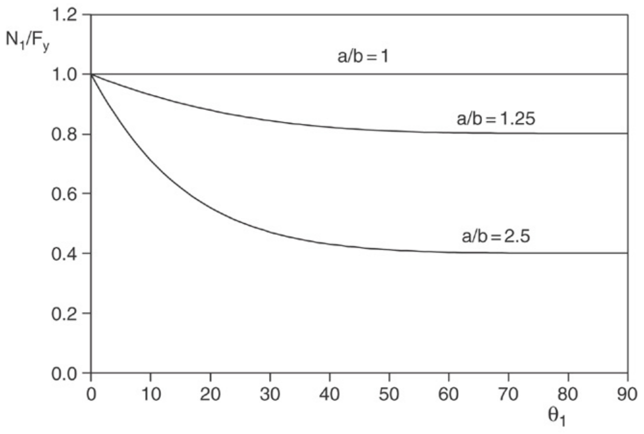

Fig. 10. Analytical dimensionless variation of axial force N_1 with the plasticization angle θ_1.

zone in the hoop (defined by means of θ_1). From the curves it emerges that with the increases in the slenderness of the cross-section (increases in the a/b ratio) more progressive plasticization occurs along the perimeter of the hoop, while in the case of circular cross-section the phenomena is almost instantaneous.

5. Conclusions

The compressive behaviour of short concrete members having elliptical cross-sections and confined with single steel hoops was investigated. The stress–strain curve in compression was evaluated using the assumption of the Mander et al. (1988) model [14], but referring to a curve intertwining with several Mander curves, each pertaining to a level of confining pressure corresponding to the current axial and lateral strain values. The analytical model proposed is based on the calculation of the equivalent confinement pressures in equilibrium with the effective stresses along the perimeter of the elliptical hoops during the loading process. These stresses are evaluated by assuming the hoops to have only axial stiffness, and including the compressibility of the concrete core. Both phases of elastic and plastic behaviour of transverse hoops were considered, also considering variation in stresses along the perimeter of the steel hoops due to the shape of the transverse cross-section. The equivalent uniform confinement pressure was calculated as a function of the effective stress values along the hoops, while the effective confinement pressure was evaluated during the loading process by using an effectiveness coefficient, taking the reduction of the effectively confined concrete core into account. Finally, the ultimate stress corresponding to formation of a complete failure plane in the concrete core was estimated on the basis of simple equilibrium considerations.

By using the proposed model the following considerations, also confirmed experimentally by recent research available in the literature, can be drawn:

– elliptical hoops provide effective lateral confinement of concrete for non-slender cross-sections;
– non-uniform confinement pressures along the perimeter of the sections develop, due to the effective distribution of stresses along the hoops;
– plasticization of hoops develops in a nonlinear manner from the centre of the major axis to the centre of the minor axis;
– ultimate stresses corresponding to the formation of an inclined failure plane increase with the volumetric ratio of the transverse steel and with an increase in the yielding stress, while they are reduced by an increase in the compressive strength of the unconfined concrete.

Finally, the analytical results were compared with the experimental values given in the literature and with the analytical results obtained with available models, showing good capability for a wide range of variables including strength of concrete, cross-section of the confined core, hoop spacing, yielding stress and volumetric ratio of the lateral reinforcements.

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