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Mechanical Equivalent of Logical Inference

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Abstract

Structural engineers usually have a solid background in mechanics but not always a good relationship with probability theory. In most cases, this is not critical because code-based design is practically probability-free [1]. It is different for those engineers who grapple with structural health monitoring (SHM), an activity in which the objective is to estimate the state of a structure from an uncertain batch of observations. A consistent framework for making inference from uncertain information is Bayesian probability theory [2].

In my work, I propose an innovative method for statistical inference based on a formal analogy between mechanics and Bayesian probabilistic models. I start from the case of single parameter estimation, based on a set of uncertain Gaussian observations. The analogy statements (Table 1) are the following:

1) the value of the parameter is represented by the position of a rigid bar with one degree of freedom;
2) an uncertain observation of the parameter is modeled as a linear elastic spring connected at one end to the bar, with stiffness equal to the observation accuracy and pre-stretch equal to the observation;
3) an observation affected by multiple sources of uncertainty is modeled as a series of springs;
4) the posterior mean value of the parameter corresponds to the position of the bar in equilibrium;
5) the posterior accuracy of the parameter corresponds to the total stiffness of the bar.

Thanks to this analogy, we can solve any inference problem with mechanical systems composed by springs in series or parallel and solvable by applying the simple rules of mechanics. Fig. 1 shows the mechanical representation of the main simple linear Gaussian inference problems, in the case of single-parameter estimation.

With N parameters to estimate, the expected values of the parameters and their standard deviations *a posteriori* are derived by calculating the potential energy of the N-degrees of freedom mechanical system that corresponds to the inference problem in exam. The stiffness matrix of the system is obtained by calculating the Hessian matrix of the potential energy, which is, in the world of logic, the N x N accuracy matrix whose inverse provides the covariance matrix of the parameters. The expected values *a posteriori* are equal to the displacements of the mechanical system in equilibrium, and we can calculate them by minimizing the potential energy.

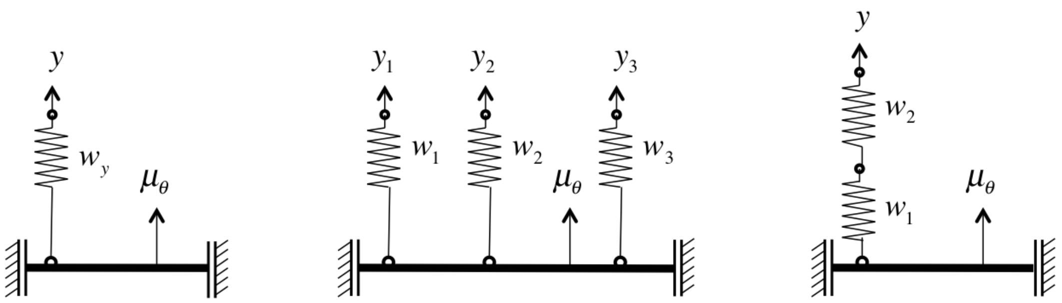
We applied this mechanical analogy to a real-life case study: the monitoring of the elongation of a cable belonging to a cable-stayed bridge—Adige Bridge [5]—which was carried out by fiber optic sensors. The aim was to estimate the intercept y_0 and the slope φ that characterize the straight line fitting the time-dependent measurements of elongation (Fig. 2a), i.e. a classical linear regression problem. In this situation, the aforementioned analogy provides a mechanical system made of a bar with two degrees of freedom—vertical translation y_0 and rotation φ—connected to linear elastic springs, which represent the elongation measurements y_i (Fig. 2b). We solved the corresponding inference problem in just a few steps by expressing the potential energy of this mechanical system with the fundamental rules of mechanics.

How does change the theory of the mechanical equivalent if we decide to involve non-Gaussian variable? As is logical, we will obtain non-linear springs, whose constitutive laws vary according to the probability distributions characterizing them. I have discovered three simple expression, which allowed me to identify the potential energy, the elastic force and the stiffness of the non-linear springs linked to the main probability distributions (Log-normal, Gumbel, Cauchy, Beta).

To sum up, by simply expressing the potential energy of the mechanical system associated to our inference scheme, my approach, allows structural engineers, with a few trivial algebraic steps, to solve any inference problem. Nevertheless, the scope of application of the method is evidently the most general, and I seek to demonstrate in the future its applicability to inference problem arising from various disciplinary fields, including cognitive science, economics and law.

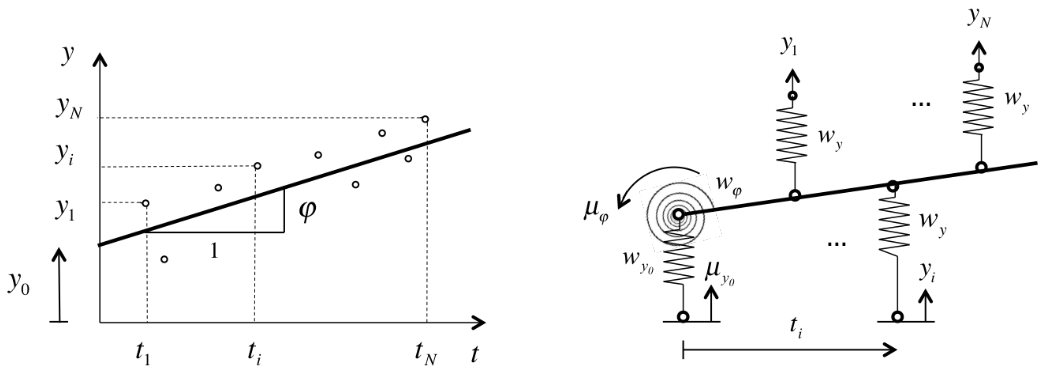
Table 1. Analogy between inference and mechanical models

Symbol	Logical meaning	Mechanical meaning
w, σ^{-2}	Accuracy, inverse-variance	Stiffness
y	Observation	Pre-stretch
μ	Expected value	Equilibrium displacement



(a) (b) (c)

Fig. 1. Mechanical analogy of simple linear Gaussian inference problems: parameter estimation based on one observation (a), three uncorrelated observations (b), one observation affected by three uncorrelated sources of uncertainty (c).



(a) World of logic (b) World of mechanics

Fig. 2. Representation of a classical linear regression problem in the world of mechanics

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1. State of the art

An analysis of the existing literature shows that very few authors in the past have developed a similar theory about the mechanical equivalent. An exception can be found in a lecture by Prof. David MacKay from the University of Cambridge on June 2006 [1], in which, he compares building a Gaussian distribution to a mechanical system of springs in series. It is also interesting to mention the work of Marko Levi [2], which shows us how to solve the main mathematical problems by using physical reasoning. As a research assistant working for Prof. Daniele Zonta (Department of Civil, Environmental and Mechanical Engineering, University of Trento) I have submitted, in collaboration with Carlo Cappello (University of Trento), some articles focused on the mechanical equivalent [3], [4], [5], in order to create a solid theoretical basis about this new approach for Civil Engineering and Structural Health Monitoring purposes. In addition, to continue this area of research, we are also preparing a scientific journal article concerning the complete theory of the mechanical equivalent [6] to be submitted by the end of this year.

In other disciplines, outside the purely mathematical or engineering, although in a few cases, it was sometimes experienced a similar approach. In this regard, I find it useful to mention as an example the researches carried out by the lawyer Carlo Bona [7], professor at the University of Law in Trento, which are aimed analytical study of legal reasoning, based on inference deductive, with aims to build a taxonomy of the typical mistakes when they fall to the judges, allowing them to be anticipated and prevented. In particular, he investigates cases in which the court must, taking the garments of the scientist, make judgments and random probability and the typical mistakes that commit performs these arguments. It is also presented the interesting hypothesis of replacing the judges with the machines, so as to avoid root in the commission of errors.

Another work, which has a lot of worth and tries to solve complex problems with a mathematical and schematic logic, is that of John Henry Wigmore (1863-1943) [8]. Wigmore was an early product of the modern, disciplinary American Law School, a model he successfully imported to Northwestern University during his length career as professor and dean of the law faculty. The backbone of Wigmore's proposal, and the core of what he taught his students, was a Chart Method for diagramming what he called "judicial proof": the justification of conclusions of fact made by the decider(s) of fact (jury or judge) at a trial. The Chart Method is a representational scheme in which lines represent what Wigmore sees as the probative processes available within the context of judicial proof. The preliminary strength of an line, assessed while the Chart is being developed, is indicated by further inflecting the arrowhead or cross: doubling it to indicate special strength, putting a "?" by it for special weakness. After the entire Chart is finished and reviewed, the constructor's final strength assessment is indicated, with a small perpendicular line indicating weakness, and a cross indicating strength. Lines connect shapes which represent what Wigmore calls "facts," by which he must mean claimed facts, or facts offered for belief. Each shape is numbered, with the number referring to a statement collected in a "key list." Each shape can be further marked to reflect its source (" " = immediately present to the jury; "' ¶ " = judicially noticed); if unmarked, it is put forward as the conclusion of some reasoning. If the fact tends to undermine the ultimate conclusion of the Chart, the base side of its shape is left off; if the fact is introduced at trial by the party opposed to the one preparing the chart, the top side of the shape is doubled.

2. Recall of probability

In this section I report same basic concepts of probability and statistic, as the use of the propagation error formulas and Bayesian probability approach, which will be useful to understand the dissertation about the mechanical equivalent of logical inference.

3. One single parameter to estimate

We start by discussing the mechanical equivalent of logical inference for single-parameter estimation. The goal is to estimate a parameter θ based on a set of uncertain information y_i. In this section I develop the complete theory of the mechanical equivalent, explaining the concept of potential energy of the mechanical system associated to the inference problem in exam. We can understand how, by expressing and kneading the potential energy, the computational cost of the parameter estimation procedures reduces. Here I apply the mechanical equivalent to the classical engineering problem of identifying the nominal resistance of a structural material (such as concrete) through direct tests and indirect tests. Direct tests yield direct observations of the unknown parameter and are, for instance, compression tests on concrete cores performed in order to identify the compressive strength, or tensile tests on steel samples when the objective is the yield strength of the steel. A test method is classified as indirect when its outcomes are only indirectly correlated to the parameter. Hardness and residual stress tests on steel, as well as pull-out tests on concrete, are typical examples of indirect methods.

4. N parameters to estimate

In this Section I extent the analogy to N parameter to estimate. The mechanical system associated to the inference problem in exam varies its degrees of freedom according to the unknown parameters to estimate, but expressing its potential energy (which will be in more variables) we can obtain the posterior standard deviations and mean values of the parameters with few algebraic steps. I get the main rules of the case, and finally I apply them to two a real-life example: the study of the deformation trend of a cable belonging to a cable-stayed bridge in Trento, with two and there parameters to estimate.

5. Non-Gaussian variables: the case of single parameter estimation

As yet the analyzed cases concerned Gaussian-variables, which are well represented by elastic linear springs. How does change the theory of the mechanical equivalent if we decide to involve non-Gaussian variables? As is logical, we will obtain non-linear springs, whose constitutive laws vary depending on the probability distributions that characterize them. As before, with the help of the elastic potential, I have formulated a general method, which allows us to define the features of each different non-linear spring (Log-normal, Gumbel, Gamma and Cauchy probability distributions). In this Section I refer only to the case of a single parameter to estimate.

6. Non-Gaussian variables: the case of N parameters to estimate

In the case of N parameters to estimate we can simply extend the same proceedings presented in previous Section. Also in this case the potential energy and the mechanical equivalent are very useful to solve a complex inference problem with non-linear probability distribution and with several parameters to estimate. The solution is carried out by using only the basic rules of the mechanics.

Conclusions

Structural engineers are usually familiar with the analysis of mechanical systems, but they are sometimes confused by applications of Bayesian probability and they prefer to make inference-using heuristics. In order to ease the approach to Bayesian inference, we have proposed a quantitative method based on a formal analogy between Gaussian probabilistic models and linear mechanics. We showed how, thanks to the proposed analogy, we could solve any complex inference problem in which the value of a single parameter or the values of more than one parameter have to be estimated based on multiple Gaussian-distributed uncertain observations, possibly correlated. We have shown that by simply expressing the potential energy of the mechanical system associated to our inference scheme we are able, with few trivial algebraic steps, to determine the posterior mean values and standard deviations of the parameters to estimate. The Potential Energy Method is therefore effective, and it is within the reach of those who do not have much confidence with the notions of probability. Its main advantage is that the method does not necessarily require the definition of the physical structure of the inference problem, allowing us to solve it with the two general steps formulated in this paper. In addition, with the aid of an example inspired by real-life scenarios, we have showed how our approach enables structural engineers to estimate, in practice, the values of the parameters that characterize the problem of the linear regression in drafting a time dependent data set. We are confident that this expeditious method of analyzing data for structural health monitoring purposes may be extended in the future to more complex inference problems, perhaps extending it to other branches of knowledge, e.g. *Economy* and *Cognitive Sciences*.

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